Energy Methods

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Contents

Variable Definitions

σ	Stress
e	Strain (normal or engineering)
F	Force
V	Shear force
P	Axial force
R	Reaction force
M	Moment
T	Torque
Q_i	General load
q_i	General displacement
p	Distributed load
b	Base dimension
h	Height dimension
r	Radius
L	Length
ΔL	Change in length
A	Area
$\mathscr{V}$	Volume
ν	Poisson's ratio
E	Young's modulus
G	Shear modulus
Q	First moment of area / statical moment of area
I	Area moment of inertia / second moment of area
J	Second polar moment of area
k_s	Linear spring constant
k_t	Torsional spring constant
Δx	Linear displacement

θ	Angular displacement
U	Strain energy
W_E	External work
K_E	Kinetic energy
H_E	Heat absorbed
Π	Total potential energy
w	Shape function
κ	Curvature

1 Introduction

Energy methods are powerful tools in structural analysis that can simplify the solving of difficult mechanical problems. Instead of using equilibrium equations, which can often result in a very difficult and complex system of equations, it is often simpler to use energy based methods. In some cases, you may have a **Statically Indeterminate** system where the internal forces and reactions are simply unknowable with equilibrium equations. For such systems, you must use tools like Castigliano's theorems or the Rayleigh-Ritz method.

The following notes will dive into an overview on structural energy and work, as well as how to use these concepts effectively to solve problems you might see when analyzing aerospace structures.

2 Strain Energy

Strain Energy (U) is the energy stored in a material under loading. Recall that materials under loading exhibit some deflection or displacement dependent on the load. When that load is removed, the strain energy stored in that displacement is released. For example, if elastically deformed, the material will return to its original unloaded state.

In a nutshell, the strain energy is the potential energy stored in material deformations that is released when the loads are removed. The general formula for strain energy is given by Eq. 1, where i and j vary from 1 to 3 and $\mathscr{V}$ is the volume of the material.

$$U = \int_{\mathscr{V}} \left(\int_0^{e_{ij}} \sigma_{ij} de_{ij} \right) d\mathscr{V} \tag{1}$$

The formula for strain energy simplifies for elastic materials, and has two possible formulations dependent on the desired variable (stress or strain). The stress formulation is given by Eq. 2, while the strain formulation is given by Eq. 3.

$$
\begin{aligned}
U = \int_{\mathscr{V}} \bigg[& \frac{1}{2E}(\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2) - \frac{\nu}{E}(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11}) \\
& + \frac{1}{2G}(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2) \bigg] d\mathscr{V}
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
U = \int_{\mathscr{V}} \bigg[& \frac{E\nu}{2(1+\nu)(1-2\nu)}(e_{11} + e_{22} + e_{33})^2 \\
& + G(e_{11}^2 + e_{22}^2 + e_{33}^2 + 2e_{12}^2 + 2e_{23}^2 + 2e_{31}^2) \bigg] d\mathscr{V}
\end{aligned} \tag{3}
$$

These equations may seem intimidating, but they are rarely used in this form thanks to additional simplifications that can be made. Some commonly used strain energy formulations will now be listed for your convenience.

1. For simple axial loading (P), the strain energy simplifies to Eq. 4.

$$
U_{axial} = \int_{\mathscr{V}} \frac{1}{2E}\sigma_{11}^2 d\mathscr{V} = \frac{1}{2}\int_0^L \frac{P(x)^2}{EA} dx \tag{4}
$$

2. For simple moment loads (M), the strain energy simplifies to Eq. 5.

$$
\begin{aligned}
U_{moment} &= \int_{\mathscr{V}} \left(\frac{1}{2E} \frac{M^2 y^2}{I^2} \right) d\mathscr{V} = \frac{1}{2}\int_0^L \frac{M(x)^2}{EI} dx \\
&= \frac{1}{2}\int \frac{M(\theta)^2}{EI} R d\theta
\end{aligned} \tag{5}
$$

3. For shear loads (V), the strain energy simplifies to Eq. 6 (Note Q here refers to the first moment of area, while Q_i in future sections refers to general loads).

$$
U_{shear} = \int_{\mathscr{V}} \left(\frac{1}{2G} \frac{V^2 Q^2}{I^2 b^2} \right) = \frac{1}{2}\int_0^L \frac{V(x)^2}{GI^2} dx \int_A \frac{Q^2}{b^2} dA \tag{6}
$$

4. For torquing loads (T), the strain energy simplifies to Eq. 7.

$$
U_{torque} = \int_{\mathscr{V}} \frac{1}{2G}(\sigma_{12}^2 + \sigma_{13}^2) d\mathscr{V} = \int_0^L \frac{T(x)^2}{2GJ} dx \tag{7}
$$

For most beam bending problems, the total strain energy is the sum of the axial, moment, and shear energies. Functions for these forces and moments can be found with shear moment diagrams. The total strain energy is given in Eq. 8. In many cases, the shear contributions are negligible compared to the moment contributions.

$$
U_{tot} = U_{axial} + U_{moment} + U_{shear} \tag{8}
$$

For beams subject to elongation, the strain energy simplifies to Eq. 9. This formula will be especially important for truss structures.

$$U_{elong} = \frac{EA}{2L}\Delta L^2 \tag{9}$$

Just as important is the energy stored in springs. The strain energy for linear and torsional springs are given by Eqs. 10 and 11, respectively.

$$U_s = \frac{1}{2}\frac{F_s^2}{k_s} = \frac{1}{2}k_s\Delta x^2 \tag{10}$$

$$U_s = \frac{1}{2}\frac{M_s^2}{k_t} = \frac{1}{2}k_t\theta^2 \tag{11}$$

Another important concept is the **Strain Energy Density** (dU), which is the strain energy per unit volume. The equation is given by Eq. 12.

$$dU = \int_0^{e_{ij}} \sigma_{ij} de_{ij} \tag{12}$$

The **Complementary Strain Energy** (U^*) does not have an easy physical interpretation. However, if you note that the strain energy is simply the area BELOW the stress-strain curve, the complementary strain energy is the area ABOVE the stress-strain curve (an example seen in Fig. 1). The governing equation for complementary strain energy is given by Eq. 13.

$$U^* = \int_{\mathscr{V}} \left(\int_0^{\sigma_{ij}} e_{ij} d\sigma_{ij} \right) d\mathscr{V} \tag{13}$$

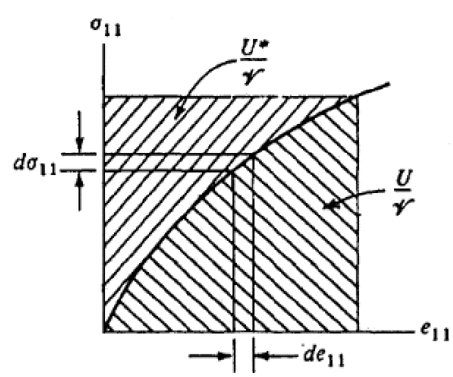

Figure 1: Strain and Complementary Strain Energy in a Stress Strain Curve for a Nonlinear Elastic Bar (Chang via Analysis of Structures)

3 External Work

External Work (W_E), otherwise known as **Work Done** on the system, is the work done on the system by external loads. In physics, you probably learned that work is load times displacement. It is no different here. The general equation for external work is given by Eq. 14, where Q_i are the loads and q_i are the corresponding displacements from said load (examples illustrated with Fig. 2). Note that this formula assumes all loads acting simultaneously. Additionally, the $\frac{1}{2}$ in the equation is due to us taking the average value of the load during the loading process (i.e. $\frac{1}{2}Q_i$).

$$W_E = \frac{1}{2}\sum_{i=1}^{n} q_i Q_i \tag{14}$$

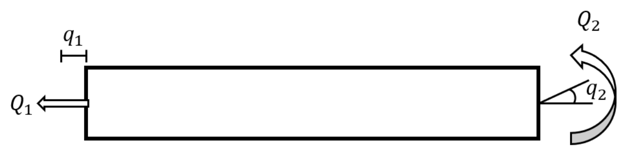

Figure 2: Generalized Loads and Displacements

Conservation of Energy allows us to relate the external work and the strain energy. According to the first law of thermodynamics, the work done to an object (W_E) plus the heat absorbed by the object (H_E) from its surroundings is equivalent to the change in internal energy of the object (ΔE). We also note that the change in internal energy of the object is a function of the kinetic (K_E) and potential (strain energy) energies. This relationship is outlined in Eq. 15.

$$\Delta E = U + K_E = W_E + H_E \tag{15}$$

For structural analysis, we will assume that the deformation process is adiabatic ($H_E = 0$) and very slow (such that equilibrium is maintained through the whole process, i.e. $K_E = 0$). Thus, we get a simplified conservation of energy equation demonstrating the relationship between work done by applied loads and internal strain energy with Eq. 16.

$$W_E = U \tag{16}$$

In most energy methods, we will need to define strain energy. Eq. 16 allows us to directly relate external work with strain energy (under a few assumptions), which is very advantageous for solving complicated problems.

4 Principles of Virtual Work and Minimum Total Potential Energy

While the principles of virtual work and minimum total potential energy are important in structural mechanics and energy methods, we will not typically directly employ them for solving problems in this course. Instead, the workhorse of energy methods are Castigliano's theorems. But before discussing Castigliano's theorems, this section on these principles will provide a deeper background on the concepts and tools that are used to derive and understand the theorems.

The **Principle of Virtual Work** is a work-energy relationship describing a structural system under **Virtual Displacements** (sometimes called a **Dummy Displacement**). Virtual displacements are displacements in a structural system that do not really exist, and therefore can be treated as arbitrary values.

The principle of virtual work states that if a structure is in equilibrium and remains as such under a virtual displacement, the external virtual work (δW_E) is equal to the change in internal strain energy (δU). This is given by Eq. 17. Note that the converse of the principle is also true (if Eq. 17, then structure is in equilibrium).

$$\delta W_E = \delta U \tag{17}$$

Note that the external virtual work can be defined using Eq. 18, and the internal virtual strain energy with Eq. 19. You may be curious why the virtual work is lacking the $\frac{1}{2}$ that the external work equations has. This is because we assume the loads to be at their final values before the virtual distortion occurs.

$$\delta W_E = \sum_{i=1}^{n} Q_i \delta q_i \tag{18}$$

$$\delta U = \int_{\mathscr{V}} \sigma_{ij} \delta e_{ij} d\mathscr{V} \tag{19}$$

The importance of the principle of virtual work is in its use as a strategy to solve for unknown loads in structural systems (statically indeterminate or otherwise). By adding a virtual displacement (typically $\delta q_i = 1$) and using the above equations, one gets a straight forward method for finding the loads.

The **Principle of Complementary Virtual Work** is another work-energy relationship describing a structural system under **Virtual Loads** (sometimes called **Dummy Loads**). Virtual loads, similarly to virtual displacements, are not real and therefore are arbitrary.

The principle of complementary virtual work states that the complementary external virtual work (δW_E^*) done by external virtual loads under the actual deformation of a structure is equal to the complementary change in internal strain energy (δU^*). This is given by Eq. 20.

$$\delta W_E^* = \delta U^* \tag{20}$$

Note that the complementary external virtual work can be defined using Eq. 21, and the complementary internal virtual strain energy with Eq. 22.

$$\delta W_E^* = \sum_{i=1}^{n} \delta Q_i q_i \tag{21}$$

$$\delta U^* = \int_{\mathscr{V}} \delta\sigma_{ij} e_{ij} d\mathscr{V} \tag{22}$$

The importance of the principle of complementary virtual work is in its use as a strategy to solve for unknown displacements in structural systems (statically indeterminate or otherwise). By adding a virtual loads (typically $\delta Q_i = 1$) and using the above equations, one gets a straight forward method for finding the displacements.

The **Total Potential Energy** of the structure (Π) is given in Eq. 23, and is the sum of external work and internal strain energy.

$$\Pi = U - W_E \tag{23}$$

The **Principle of Minimum Potential Energy** is given by Eq. 24, where $\delta\Pi$ is the variation of the total potential energy. This states that of all the displacement fields which satisfy the system constraints, the correct state is that which makes the total potential energy of the structure a minimum. In other words, virtual displacements will not change the total potential energy because the structure will be at a minimum energy.

$$\delta\Pi = 0 \quad \forall \delta q_i \tag{24}$$

The **Total Complementary Potential Energy** of the structure (Π^*) is given in Eq. 25, and is the sum of complementary external work and complementary internal strain energy.

$$\Pi^* = U^* - W_E^* \tag{25}$$

The **Principle of Minimum Complementary Potential Energy** is given by Eq. 26, where $\delta\Pi^*$ is the variation of the total complementary potential energy. This states that of all the stress states which satisfy the system constraints, the correct state is that which makes the total complementary potential energy of the structure a minimum. In other words, virtual loads will not change the total complementary potential energy because the structure will be at a minimum energy.

$$\delta\Pi^* = 0 \quad \forall \delta Q_i \tag{26}$$

5 Castigliano's 1st Theorem

As stated previously, the principles of virtual work and minimum total potential energy are important for deriving and understanding Castigliano's theorems. Applying Castigliano's theorems is the bread and butter of many energy methods courses. Understanding what they are and how to use them will allow you to use energy methods to solve previously unsolvable problems in structural mechanics.

Castigliano's 1st Theorem states that if the strain energy stored in an elastic structure is expressed as a function of the generalized displacements, the first partial derivative of the strain energy with respect to any of the generalized displacements (q_i) is equal to the corresponding generalized load (Q_i). This is described formulaically with Eq. 27.

In a nutshell, Castigliano's 1st theorem allows us to solve for the load given knowledge of the strain energy of the structure and the measured displacements. Applying this theorem can appear tricky at first, but by breaking it down into manageable steps you will find that it becomes very simple. In many cases, even simpler than applying equilibrium equations.

Applying Castigliano's 1st Theorem:

1. Place displacements (q_i) at all non-fixed locations with applied external loads acting in the direction of that load
2. Find the strain energy of the system (U)
3. Rewrite the strain energy of the system in terms of the displacements ($U(q_i)$)
4. Solve the derivative equation given by Eq. 27 for the unknown load(s) (Q_i)

$$\frac{\partial U}{\partial q_i} = Q_i \tag{27}$$

Let's demonstrate this process with a simple example. Given a truss structure like that depicted in Fig. 3, we would like to find the value for a load Q_1 and the internal axial loads on the bars (F_A and F_B) given a displacement q_1.

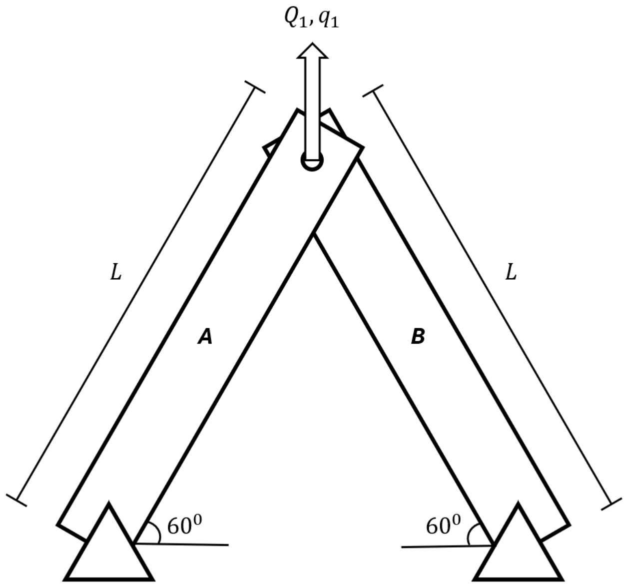

Figure 3: Castigliano's 1st Theorem Example

The first step is to place non-fixed displacements in our structure, which is already done for us in Fig. 3.

The second step is to find the strain energy of the system. We have 2 beams, A and B. The energy in each beam can be summed to find the strain energy of the entire system. This is given by Eq. 28.

$$U = U_A + U_B \tag{28}$$

An important aspect to consider is that A and B will have elongations given by ΔL_A and ΔL_B respectively (thanks to Q_1). These elongations are pictured in Fig. 4.

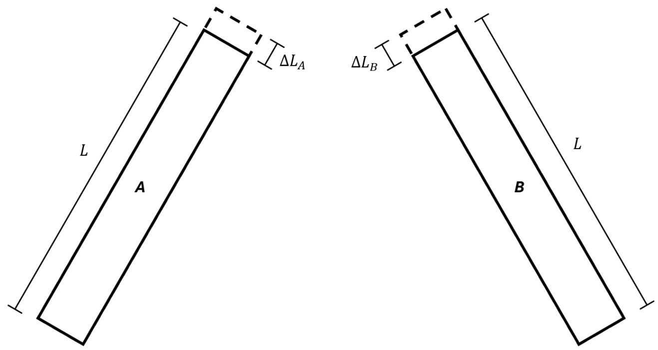

Figure 4: Beam Elongations

From the previous section on strain energy, we can rewrite the strain energy of the system with Eq. 29 (assuming the cross sectional area A and Young's modulus E are the same for both beams).

$$U = \frac{EA}{2L}\Delta L_A^2 + \frac{EA}{2L}\Delta L_B^2 = \frac{EA}{2L}(\Delta L_A^2 + \Delta L_B^2) \tag{29}$$

The third step is to rewrite the strain energy in terms of the displacement q_1. This can be done by relating the elongations ΔL_A and ΔL_B to q_1 using transformations as seen in Fig. 5, and are given by Eq. 30.

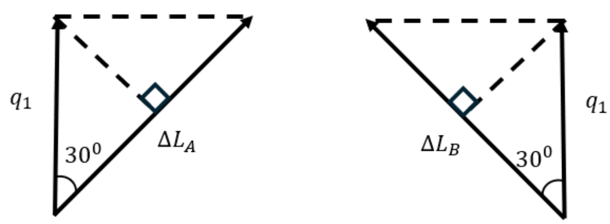

Figure 5: Elongations and Displacements

$$\begin{aligned} \Delta L_A &= \cos(30^\circ) q_1 \\ \Delta L_B &= \cos(30^\circ) q_1 \end{aligned} \tag{30}$$

The new strain energy equation in terms of q_1 can now be given with Eq. 31.

$$U = \frac{EA}{L}\cos^2(30°)q_1^2 \tag{31}$$

The final step is to use Eq. 27 to relate the strain in terms of displacement to the unknown load Q_1. This is accomplished with Eq. 32. And with that, we've solved for the unknown load!

$$Q_1 = \frac{\partial U}{\partial q_1} = 2\frac{EA}{L}\cos^2(30°)q_1 \tag{32}$$

We also would like to solve for the internal axial loads in the bars. This can be easily accomplished using elastic stress-strain relationships. We can represent the internal axial load of the beam using the equation derived in Eq. 33.

$$\begin{aligned} \sigma &= E\varepsilon \\ \frac{F}{A} &= E\frac{\Delta L}{L} \\ F &= EA\frac{\Delta L}{L} \end{aligned} \tag{33}$$

Implementing these relationships, we find the internal axial loads given by Eq. 34.

$$F_A = F_B = EA\frac{\cos(30°)q_1}{L} \tag{34}$$

6 Castigliano's 2nd Theorem

Castigliano's 2nd Theorem states that if the strain energy stored in an elastic structure is expressed as a function of the generalized loads, the first partial derivative of the strain energy with respect to any of the generalized loads (Q_i) is equal to the corresponding generalized displacement (q_i). This is described formulaically with Eq. 36.

In a nutshell, Castigliano's 2nd theorem allows us to solve for the displacement of a structure given knowledge of the strain energy and the applied loads. Applying this theorem can appear tricky at first, but by breaking it down into manageable steps you will find that it is mostly bark and no bite.

Applying Castigliano's 2nd Theorem:

1. If necessary, add an imaginary load ($Q_{imag} = 0$) at the location of the desired deflection
2. If the structure is statically indeterminate, select redundant loads (Q_{red} with corresponding displacement q_{red}) at kinematically constrained locations (fixed with a joint or spring) until the structure is no longer statically indeterminate
3. Find the strain energy of the system (U) in terms of the known external loads, imaginary loads, and redundant loads
4. Solve for the redundant loads using Castigliano's 2nd theorem applied in Eq. 35 (note generally $q_{red} = 0$ except for springs)

$$\frac{\partial U}{\partial Q_{red}} = q_{red} \tag{35}$$

5. Using the newly determined values of the redundant loads, find strain energy of the system in terms of known external loads and imaginary loads only
6. Solve for the desired displacement using Castigliano's 2nd theorem with Eq. 36. Use $Q_{imag} = 0$ if necessary

$$\frac{\partial U}{\partial Q_i} = q_i \tag{36}$$

This process is certainly a bit more complicated than Castigliano's 1st theorem. To more clearly illustrate this process, let's do a simple example. Given a beam structure like that depicted in Fig. 6, we would like to find the value for a displacement q_2 given an applied load Q_1.

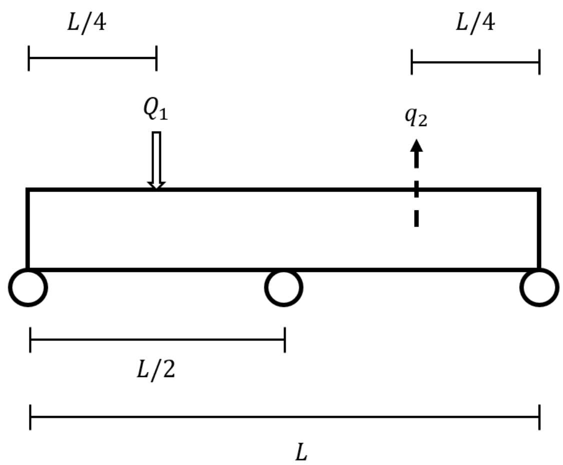

Figure 6: Castigliano's 2nd Theorem Example

The first step is to add an imaginary load if necessary. In this case, an imaginary load is necessary since the displacement we are solving for (q_2) does not have an associated load (what can we take the derivative with respect to). Let's add an imaginary load $Q_2 = 0$ to the beam like seen in Fig. 7.

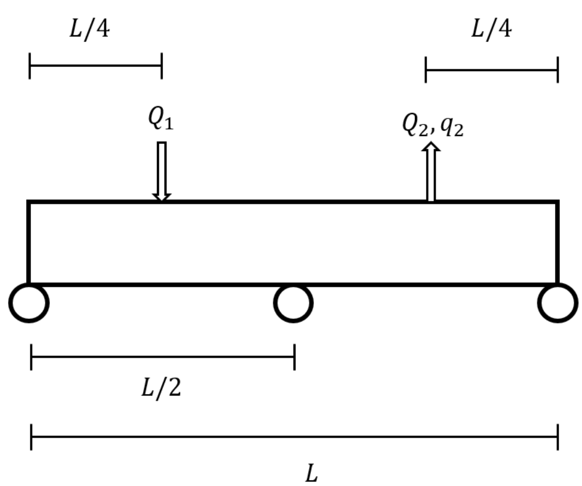

Figure 7: Adding Imaginary Load Q_2

The second step is to fully characterize all external applied loads on the structure. For this, we will need to use the free body diagram shown in Fig. 8.

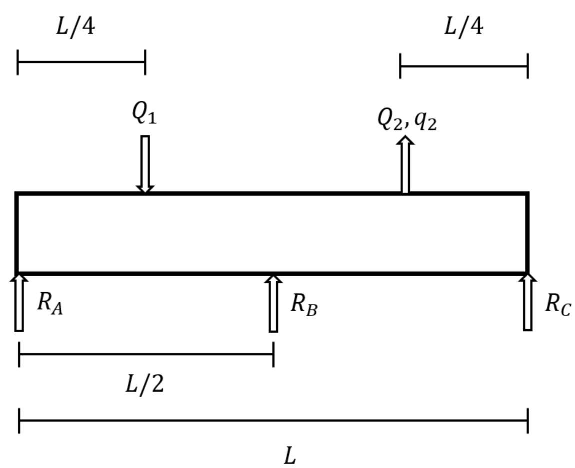

Figure 8: Free Body Diagram

We can use the equilibrium equations to solve for the boundary condition reaction forces (R). These are given by Eqs. 37 and 38. Note that there are no forces in the x direction, so that equilibrium equation is of no use to us.

$$\sum F_y = 0 = R_A - Q_1 + R_B + Q_2 + R_C \tag{37}$$

$$\sum M_A = 0 = -Q_1 \frac{L}{4} + R_B \frac{L}{2} + Q_2 \frac{3L}{4} + R_C L \tag{38}$$

You may notice we have an immediate problem. This structure is statically indeterminate! There are 3 unknown forces (R_A, R_B, and R_C) and only 2 equilibrium equations. Because of this, we need to introduce a redundant load, or a load we take as known. The load should be taken at a kinematically constrained location (a fixed joint or a spring). Additionally, appropriately selected redundant loads, if removed from the structure, should leave the structure as statically determinate. Let's choose R_C to be our redundant force. To verify this is a good choice, let's sketch the structure without R_C (Fig. 9) and ensure it is statically determinate. As we can see with Eqs. 39 and 40, we have two equations and two unknowns. Thus, the structure is statically determinate! So, R_C is a good choice as a redundant load.

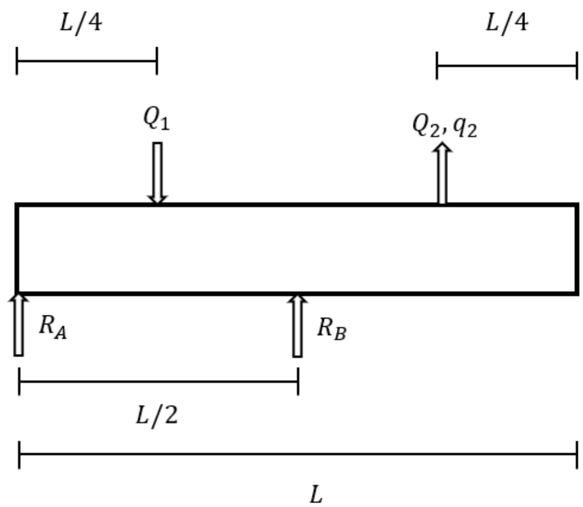

Figure 9: FBD without Redundant Load R_C

$$\sum F_y = 0 = R_A - Q_1 + R_B + Q_2 \tag{39}$$

$$\sum M_A = 0 = -Q_1\frac{L}{4} + R_B\frac{L}{2} + Q_2\frac{3L}{4} \tag{40}$$

Taking R_C as the redundant load, we now have only two unknowns (R_A and R_B) and 2 equations, which is solvable. This results in Eqs. 41 and 42.

$$R_A = \frac{Q_1}{2} + \frac{Q_2}{2} + R_C \tag{41}$$

$$R_B = \frac{Q_1}{2} - \frac{3Q_2}{2} - 2R_C \tag{42}$$

The third step is to determine the strain energy in the system in terms of the known external loads, imaginary loads, and redundant loads. In our case, these are Q_1, Q_2, and R_C. The equation for strain energy in this beam is given by Eq. 43. Note that there are no axial or torsional loads, and we assume the shear contribution is negligible in this case.

$$U = \frac{1}{2}\int_0^L \frac{M(x)^2}{EI}dx \tag{43}$$

To find the internal bending moment $M(x)$, we will need to follow the shear and moment diagram cutting procedure. An example of the first cut is given by Fig. 10.

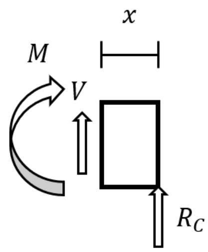

Figure 10: 1st Shear and Moment Cut

At the end of the cutting process, we get the strain energy given in Eq. 44.

$$U = \frac{L^3(Q_1^2 + 3Q_1Q_2 + 6Q_1R_C + 6Q_2^2 + 26Q_2R_C + 32R_C^2)}{768EI} \tag{44}$$

The next step is to solve for the redundant loads (R_C). We take advantage of the fact that the redundant loads are kinematically constrained, and the displacement of a load fixed with a joint will be zero. The equation then is given by Eq. 45.

$$\frac{\partial U}{\partial R_C} = \frac{L^3(6Q_1 + 26Q_2 + 64R_C)}{768EI} = 0 \tag{45}$$

After solving, we find that the redundant load is equivalent to Eq. 46.

$$R_C = -\frac{3Q_1 + 13Q_2}{32} \tag{46}$$

We can then substitute this value into our strain energy to get it in terms of known and imaginary loads only. This is given by Eq. 47.

$$U = \frac{L^3(23Q_1^2 + 18Q_1Q_2 + 23Q_2^2)}{24576EI} \tag{47}$$

Finally, we have all the information we need to solve for the displacement q_2. To do this, we use Castigliano's 2nd theorem at the corresponding load, which for us is the imaginary load Q_2. This is shown in Eq. 48.

$$q_2 = \frac{\partial U}{\partial Q_2} = \frac{L^3(18Q_1 + 46Q_2)}{24576EI} \tag{48}$$

Last, we recall that the imaginary load is imaginary of course ($Q_2 = 0$)! After substituting this into Eq. 48, we find the displacement given by Eq. 49. And that concludes our example.

$$q_2 = \frac{3L^3Q_1}{4096EI} \tag{49}$$

7 Rayleigh-Ritz Method

While Castigliano's theorems provide exact solutions to solving for unknown loads and displacements in structural systems, we can also use approximate methods to estimate these unknowns more efficiently.

The most popular approximation technique is the **Rayleigh-Ritz Method**. The Rayleigh-Ritz method approximates the displacement field with simple **Shape Functions** ($w(x)$) and a finite number of coefficients. The shape functions are based off of the kinematic boundary constraints. Deriving these shape functions is not essential for this class, so I will not go into detail here. It is safe to assume that for all Rayleigh-Ritz method problems you will be given the shape functions necessary.

It is, however, vital to really understand what the shape function is. At any point x in a 1D structure, the displacement is given by the shape function $w(x)$. The angular deflection is equal to the slope of the beam given by $\frac{\partial w(x)}{\partial x}$. The curvature ($\kappa$) of the beam at x is given by $\frac{\partial^2 w(x)}{\partial x^2}$. Note that the curvature can be related to the bending moment via Eq. 50.

$$M = -EI\kappa \tag{50}$$

Applying Rayleigh-Ritz Method:

1. Get the shape function $w(x)$ (typically given)

2. Find the total potential energy using Eq. 51. The strain energy (U) of a bending beam is typically given by Eq. 52. The external work is typically given by some combination of Eqs. 53 (work of a distributed load), 54 (work of a point load), and 55 (work of an applied moment)

$$\Pi = U - W_E \tag{51}$$

$$U = \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 w(x)}{\partial x^2}\right)^2 dx \tag{52}$$

$$W_E = \int_0^L Q_i(x)w(x)dx \tag{53}$$

$$W_E = Q_i(x)w(x) \tag{54}$$

$$W_E = Q_i(x)\frac{\partial w(x)}{\partial x} \tag{55}$$

3. Set the derivative of the total potential energy with respect to each unknown coefficient to zero and solve for each coefficient as shown in Eq. 56

$$\frac{\partial \Pi}{\partial c_1} = 0, \quad \frac{\partial \Pi}{\partial c_2} = 0, \quad ... \quad \frac{\partial \Pi}{\partial c_n} = 0 \tag{56}$$

4. Substitute coefficients in the shape function

5. Solve for the external work, strain energy, total potential energy, displacement, or whatever else is required

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Again, let us use an example to illustrate this process. Our example is given by Fig. 11, which is a beam under a point load. Our shape function is given by Eq. 57, where we have an unknown coefficient c_1. Our goal will be to identify the displacement at $L/2$.

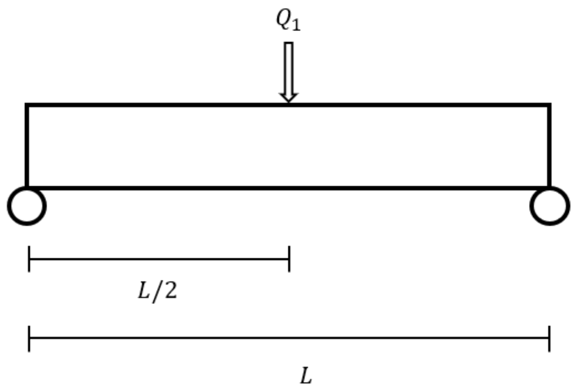

Figure 11: Rayleigh-Ritz Method Example

$$w(x) = c_1 x^2 \tag{57}$$

The first step is to get our shape function. Thankfully, this is provided in the problem via Eq. 57.

The second step is to determine the total potential energy (Π). As seen in Eq. 51, the total potential energy is made up of two components: the strain energy (U) and the external work (W_E). Solving for these using previously noted equations, we find the total potential energy given by Eq. 58.

$$\Pi = \frac{8c_1^2 EIL - Q_1 c_1 L^2}{4} \tag{58}$$

The third step is to solve for the unknown coefficient (c_1) using the derivative of the total potential energy with respect to each unknown coefficient (Eq. 59).

$$\frac{\partial \Pi}{\partial c_1} = 16c_1 EIL - Q_1 L^2 = 0 \tag{59}$$

The value for our unknown coefficient is then solved to be Eq. 60.

$$c_1 = \frac{Q_1 L}{16EI} \tag{60}$$

We can now plug this value back into our shape function, given by Eq. 57.

$$w(x) = \frac{Q_1 L}{16EI} x^2 \tag{61}$$

Finally, we can find the displacement (q_1) caused by Q_1 at $L/2$ by plugging $x = L/2$ into our shape function Eq. 61. The result is given in Eq. 62.

$$q_1 = \frac{Q_1 L^3}{64EI} \tag{62}$$

A MATLAB Codes

```
% AA240
% Energy Methods Code - Castigliano's 2nd Theorem
% Author(s): Mark Paral

% Clear workspace
clc
clear all
close all

% Create symbolic variables
syms E I L RC Q1 Q2 x q2
RB = Q1/2 - 3*Q2/2 - 2*RC;

% Create strain energy
U = 0;

% Strain energy x = [0,L/4]
M1 = RC*x;
U1 = 1 / (2*E*I) * int(M1^2,x,[0,L/4]);

% Strain energy x = [L/4,L/2]
M2 = RC*x + Q2*(x-L/4);
U2 = 1 / (2*E*I) * int(M2^2,x,[L/4,L/2]);

% Strain energy x = [L/2,3L/4]
M3 = RC*x + Q2*(x-L/4) + RB*(x-L/2);
U3 = 1 / (2*E*I) * int(M3^2,x,[L/2,3*L/4]);

% Strain energy x = [3L/4,L]
M4 = RC*x + Q2*(x-L/4) + RB*(x-L/2) - Q1*(x-3*L/4);
U4 = 1 / (2*E*I) * int(M4^2,x,[3*L/4,L]);

% Total strain energy
U = simplify(U1 + U2 + U3 + U4);
pretty(U)

% Calculate redundant load
eq1 = 0 == diff(U,RC);
pretty(eq1)
RC_val = simplify(solve(eq1,RC));
pretty(RC_val)

% Sub in redundant load to strain energy
```

```
U = simplify(subs(U,RC,RC_val));
pretty(U)

% Calculate displacement
eq2 = q2 == diff(U,Q2);
pretty((eq2));
q2_val = simplify(subs(solve(eq2,q2),Q2,0));
pretty(q2_val)
```