Fundamentals of Solid Mechanics

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Contents

Variable Definitions

F	Force
V	Shear force
M	Moment
T	Torque
b	Base dimension
h	Height dimension
r	Radius
L	Length
ΔL	Change in length
A	Area
σ	Stress
τ	Shear stress
$\bar{\sigma}$	Stress vector
$\boldsymbol{\sigma}$	Stress tensor
$\bar{T}$	Traction vector
ε	Strain
γ	Engineering strain
$\bar{e}$	Strain vector
$\boldsymbol{e}$	Strain tensor
ν	Poisson's ratio
E	Young's modulus
G	Shear modulus
$\boldsymbol{R}$	Direction cosine matrix (DCM)
$\boldsymbol{T}$	Transformation matrix
Q	First moment of area / statical moment of area
I	Area moment of inertia / second moment of area
J	Second polar moment of area

$\boldsymbol{C}$ Stiffness matrix

$\boldsymbol{S}$ Compliance matrix

α Angular acceleration

ϕ Angle of twist

1 Introduction

Solid Mechanics is the study of how solid materials will behave under a variety of forces and deformations. It's an important topic for aerospace engineering as we want to predict how our materials will hold up under different loading conditions, say on the wing of an aircraft. This topic underpins all of aerospace structures, so learning and understanding it to a high degree is essential to be successful in this or any aerospace structures related class.

With that motivation, let's dive into the fundamental concepts you will need to succeed in a graduate level aerospace structures course.

2 Isotropic Materials

Isotropic Materials are materials who's properties remain the same independent of direction. In essence, the application of loads in any direction will produce predictable and uniform responses. This allows us to simplify many of the calculations analyzing performance. Note that only isotropic materials are generally considered in fundamental structural mechanics courses. Examples of isotropic materials include many metals and plastics.

There are many other classifications for materials that are not isotropic. If interested, the reader may review

1. Quasi-Isotropic Materials (sometimes called Transversely Isotropic)
2. Anisotropic Materials
3. Orthotropic Materials
4. Homogeneous Materials
5. Heterogeneous Materials

3 Stress and Strain

The most fundamental quantities in solid mechanics are stress and strain. But what exactly do these concepts represent and why are they important?

3.1 Stress

Stress (typically denoted as σ) is the force per unit area acting on an object (Eq. 1). The units are typically $\left[\frac{N}{m^2}\right]$ in metric.

$$\sigma = \frac{F}{A} \tag{1}$$

Stress is meaningful as it illustrates how the forces are distributed across the material. With that information, we can evaluate how the material performs under different loading conditions.

There are two categories of stress: normal and shear. Differentiating between the two is a matter of direction. **Normal Stress** (σ) acts in a direction perpendicular to the surface of a material. **Shear Stress** (τ) acts in a direction parallel to the surface of a material. This is more clearly depicted in Fig. 1. An important aspect of shear stresses is that all shear stresses $\tau_{ij} = \tau_{ji}$ as unbalanced shear stresses would cause a net torque (impossible for an internal quantity).

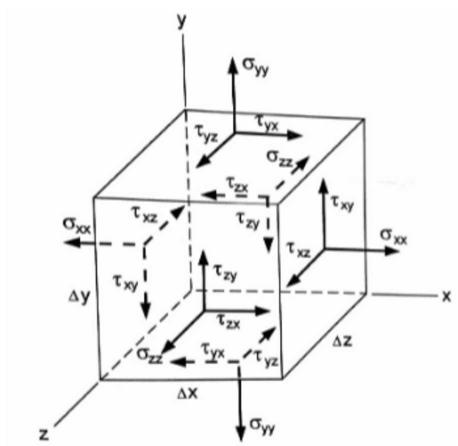

Figure 1: Normal and Shear Stress (Chang via Analysis of Structures)

There are two primary of ways of representing the stresses acting on a material: vector and tensor. The **Stress Vector** is simply a vector of all the stresses acting on a material. An example of this is given by Eq. 2 (recall $\tau_{ij} = \tau_{ji}$). The **Stress Tensor** is a matrix of all the stresses acting on a material. An example of this is given by Eq. 3.

$$\bar{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix} \tag{2}$$

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \tag{3}$$

The **Principle Stress** is the stress in the particular case where the shear stresses are zero ($\tau_{ij} = 0$). This can be accomplished for any set of stresses through a frame rotation. The process for finding principle stresses is a matter of eigenvalues and eigenvectors. The eigenvalues are the principle stresses, while the corresponding eigenvectors are the directions those principle stresses are acting in (relative to the original frame of reference). Note that by convention, $\sigma_1 > \sigma_2 > \sigma_3$.

Traction ($\bar{T}$) is the force per area acting on a specific surface of a material. In essence, it is the stress acting on a specific plane in the material. While the stress is a generalized representation for loading at a point in a material, the traction will give specifics on a plane defined by the unit normal vector $\hat{n}$. The equation for finding the traction vector is given in Eq. 4. The normal and shear components of the traction vector are given by Eqs. 5 and 6, respectively.

$$\bar{T} = \boldsymbol{\sigma}\hat{n} \tag{4}$$

$$|\bar{T}_{normal}| = \bar{T} \cdot \hat{n} \tag{5}$$

$$|\bar{T}_{shear}| = \|\bar{T} \times \hat{n}\|_2 \tag{6}$$

3.2 Strain

Strain (typically denoted as ε) is a measure of the deformation of a material under stress (Eq. 7). Strain is a unit-less quantity.

$$\varepsilon = \frac{\Delta L}{L} \tag{7}$$

Strain is meaningful as it illustrates how the material changes shape under loading conditions described by stress (see these fundamental quantities are already becoming important!).

There are two categories of strain: normal and shear (just as with stress). The differences between these three are a matter of stress applied. **Normal Strain** is the strain in response to a normal stress. **Shear Strain** is the strain in response to a shear stress. These strains are depicted in Fig. 2.

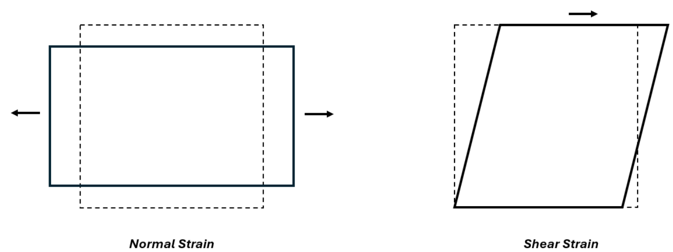

Figure 2: Normal and Shear Strain

The **Engineering Strain** (γ) is an important quantity in solid mechanics. It is defined as being twice the strain (Eq. 8). While we don't use this quantity for normal strains, it is almost exclusively used instead of shear strain for most equations, so be aware!

$$\gamma = 2\varepsilon \tag{8}$$

There are, similar to stresses, two primary of ways of representing the strains acting on a material: vector and tensor. The **Strain Vector** is simply a vector of all the strains of a material. An example of this is given by Eq. 9 (recall $\tau_{ij} = \tau_{ji}$, similarly $\gamma_{ij} = \gamma_{ji}$). The **Strain Tensor** is a matrix of all the Strains acting on a material. An example of this is given by Eq. 10.

$$\bar{e} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix} \tag{9}$$

$$\boldsymbol{e} = \begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_{zz} \end{bmatrix} \tag{10}$$

The **Principle Strain** is the case where the shear strains are zero ($\gamma_{ij} = 0$). This can be accomplished for any set of strains through a frame rotation. The process for finding principle strains is a matter of eigenvalues and eigenvectors. The eigenvalues are the principle strains, while the corresponding eigenvectors are the directions those principle strains are acting in (relative to the original frame of reference). Note that by convention, $\varepsilon_1 > \varepsilon_2 > \varepsilon_3$.

Poisson's Ratio (ν) is a quantity that relates the strain in the transverse direction (perpendicular to loading) to the strain in the axial direction (parallel to loading). It is defined in Eq. 11, and the directions are illustrated in Fig. 3.

$$\nu = -\frac{\varepsilon_{trans}}{\varepsilon_{axial}} \tag{11}$$

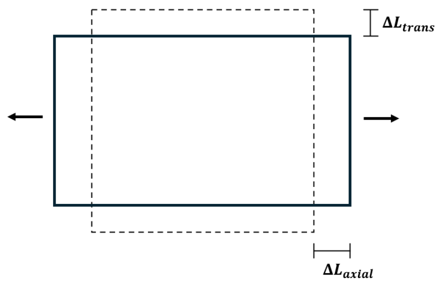

Figure 3: Poisson's Ratio Relationship

4 Frame Rotations

In many cases, it may be advantageous to perform mechanical calculations in a frame different from the one initially given in a problem (think principle stresses, for example). Thus, being able to transform from one frame to another is an important skill to have. An example of a frame rotation is depicted in Fig. 4.

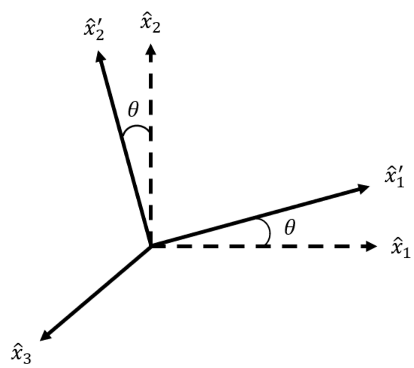

Figure 4: Sample Frame Rotation about $\hat{x}_3$

A type of transformation matrix you're probably familiar with is the **Direction Cosine Matrix** (**R**), which describes the rotation of a vector from one frame to another (Eq. 12). This matrix can be used to transform stress tensors with Eq. 13.

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \boldsymbol{R} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{12}$$

$$\boldsymbol{\sigma}' = \boldsymbol{R}\boldsymbol{\sigma}\boldsymbol{R}^\top \tag{13}$$

An example of a direction cosine matrix is given in Eq. 14, where there is a rotation θ about the $\hat{x}_3$ axis (like that depicted in Fig. 4).

$$R_{x_3} = \begin{bmatrix} cos(\theta) & sin(\theta) & 0 \\ -sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{14}$$

A **Transformation Matrix** ($\boldsymbol{T}$) is a matrix used to rotate the stress vector from one frame to another (Eq. 15). The transformation matrix can be determined by solving for each component of the stress tensor (σ'_{ij}) with the direction cosine matrix method, then rewriting in vector form.

$$\bar{\sigma}' = \boldsymbol{T}\bar{\sigma} \tag{15}$$

5 Area Moments

The area moments are important to define as they give some quantifiable value to the concepts of bending and torsion resistance.

5.1 First Moment of Area

The **First Moment of Area** or **Statical Moment of Area** (Q) is a value describing the distribution of the area of a shape in relation to the axes. The equations describing this quantity are given by Eqs. 16 and 17. Note that for these equations and figures, we assume $\hat{x}$ and $\hat{y}$ are axes that make up the cross sectional plane.

$$Q_x = \int y dA \tag{16}$$

$$Q_y = \int x dA \tag{17}$$

This concept is useful for calculating the centroid of a shape as well as determining the shear distribution across a cross-section (more on this in future sections).

5.2 Area Moment of Inertia / Second Moment of Area

The **Area Moment of Inertia** or **Second Moment of Area** (I) is a value describing how resistant to bending a shape is. This is the quantity that justifies an I-beam is a more efficient design than a solid rectangle, for example. The general equations for describing the area moment of inertia are given by Eqs. 18 and 19. A few specific cases are given in Tab. 1. Note that for these equations and figures, we assume $\hat{x}$ and $\hat{y}$ are axes that make up the cross sectional plane.

$$I_x = \int y^2 dA \tag{18}$$

$$I_y = \int x^2 dA \tag{19}$$

Cross-Section	**Area Moment of Inertia**
Solid Rectangle	$I_x = \frac{bh^3}{12}$, $I_y = \frac{b^3h}{12}$
Solid Circle	$I_x = I_y = \frac{\pi}{4}r^4$
Hollow Circle	$I_x = I_y = \frac{\pi}{4}(r_{outer}^4 - r_{inner}^4)$

Table 1: Area Moment of Inertia for Select Cross-Sections Centered at Origin

An important element to remember when calculating the Area Moment of Inertia is you can add and subtract simpler moments to get the more complex geometries. For example, the hollow circle moment can be derived by taking a solid circle at the outer radius and subtracting a solid circle at the inner radius.

The **Parallel Axis Theorem** allows you to use the area moment of inertia of a cross section centered at the origin to find one translated some distance d along an axis. The governing equation is given by Eq. 20, and a demonstration is shown in Fig. 5.

$$I_x = I_{cx} + Ad^2 \tag{20}$$

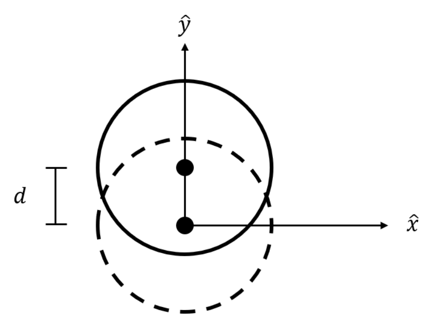

Figure 5: Parallel Axis Theorem Application

5.3 Second Polar Moment of Area

The **Second Polar Moment of Area** (J) is a value that describes how resistant the object is to torsion. The equation describing this quantity is given by

Eq. 21, and is depicted in Fig. 6. Note that we assume that $\hat{z}$ is the twisting axis.

$$J = \int \rho^2 dA \tag{21}$$

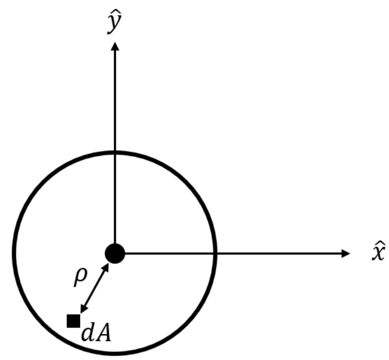

Figure 6: Second Polar Moment of Area

The **Perpendicular Axis Theorem** allows you to calculate the second polar moment of inertia using the area moment of inertia, which is often more straight forward to compute. The equation for the perpendicular axis theorem is given by Eq. 22.

$$J = I_x + I_y \tag{22}$$

6 Hooke's Law

In the application of solid mechanics, we may have situations where we know the strain and desire the stress value or vice versa. Thankfully, there is an equation which relates the two. This equation is known as **Hooke's Law**, and it describes the linear relationship between elastic materials. Hooke's law is defined in Eqs. 23, 24, 25, and 26.

Eq. 23 defines Hooke's law for normal stresses and strains. The E value is commonly referred to as the **Young's Modulus** and relates stress and strain linearly. Fig. 7 depicts this relationship well.

$$\sigma = E\varepsilon \tag{23}$$

Eq. 24 defines Hooke's law for shear stresses and strains. The G value is commonly referred to as the **Shear Modulus** and relates stress and engineering strain linearly.

$$\tau = G\gamma \tag{24}$$

Eqs. 25 and 26 define Hooke's law in general for both normal and shear stresses and strains. The $\boldsymbol{C}$ value is commonly referred to as the **Stiffness Matrix** and the $\boldsymbol{S}$ value is the **Compliance Matrix** (Yes, you read that right. It's confusing I know). As you may have guessed, these matrices can be built using Young's and shear moduli, as well as other material properties.

$$\bar{\sigma} = \boldsymbol{C}\bar{e} \tag{25}$$

$$\bar{e} = \boldsymbol{S}\bar{\sigma} \tag{26}$$

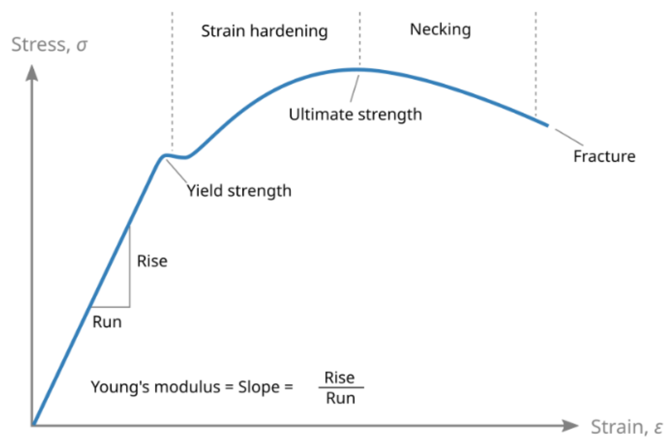

Figure 7: Stress-Strain Relationship (Nicoguaro via Wikimedia)

Taking a longer look at Fig. 7, you will notice that initially the stress-strain relationship is linear. In this region, Hooke's law is accurate and applicable. This is the region known as **Elastic Deformation**, where any deformations made are not permanent and the material will return to its original unloaded state once the load is removed. The **Yield Strength** marks the transition from elastic deformation to **Inelastic Deformation** or **Plastic Deformation**. In this region, even once the load is removed the material will not return to its original unloaded state. The material will now have a permanent deformation.

Revisiting the elastic constants (Young's modulus, shear modulus, and Poisson's ratio), there is a vital equation that relates the three together for isotropic material given by Eq. 27.

$$E = 2G(1 + \nu) \tag{27}$$

7 Equilibrium Equations

Some of the most fundamental and important equations in solid mechanics are the **Equilibrium Equations**. These equations balance the forces and moments inside a material body which is in equilibrium, or not accelerating (typically at rest or in constant motion). The equations are described by Eqs. 28 (sum of the forces in any direction are zero) and 29 (sum of the moments in any direction are zero).

$$\sum F_{x,y,z} = 0 \tag{28}$$

$$\sum M_{x,y,z} = 0 \tag{29}$$

8 Beam Bending and Torsion

8.1 Beam Bending

One of the most fundamental loading situations in mechanics is **Beam Bending**. Beam bending can occur in many different scenarios, from applied moments to triaxial loading. Importantly, there are stress equations that can be applied for isotropic material under bending (like seen in Fig. 8).

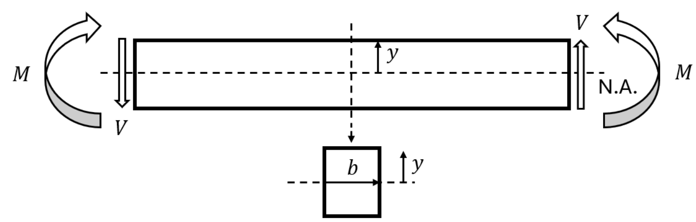

Figure 8: Beam Bending Example

First you have Eq. 30, which describes the axial stress from a bending moment. It is dependent on the bending moment (M), the second moment of area (I), and the location offset from the neutral axis (y). Note the **Neutral Axis** is the axis about which there are no longitudinal stresses or strains in the structural element.

$$\sigma = \frac{My}{I} \tag{30}$$

There is also Eq. 31, which describes the shear stress from a bending moment. It is dependent on the shear force (V), the first moment of area (Q), the second moment of area (I), and the width of the cross section (b).

$$\tau = \frac{VQ}{Ib} \tag{31}$$

8.2 Shaft Torsion

Another fundamental loading situation is **Shaft Torsion**. Torsion occurs when a material is twisted (by twisting both ends in opposite directions or by holding one end steady and twisting the other). It occurs with an applied torque, and an example is depicted in Fig. 9.

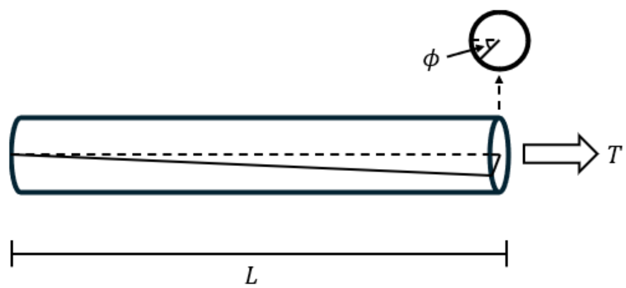

Figure 9: Shaft Torsion Example

It causes shear stresses within the material which can be calculated using Eq. 32 (T is applied torque, r is radius, and J is the second polar moment of area). Also note that the maximum shear stress in a shaft under torsion occurs at the surface where the radius is at a maximum.

$$\tau = \frac{Tr}{J} \tag{32}$$

Another important relationship is between torque, second moment of area (I), and the angular acceleration (α) given in Eq. 33.

$$T = I\alpha \tag{33}$$

Often it is useful to know the amount of twist a shaft undergoes given a torque. This can be found using the equation for angle of twist (ϕ) given by Eq. 34, where L is the shaft length and G is the shear modulus.

$$\phi = \frac{TL}{GJ} \tag{34}$$