Structural Failure

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Contents

Variable Definitions

F	Force
V	Shear force
P	Axial force
R	Reaction force
M	Moment
T	Torque
σ	Stress
ε	Strain
L	Length
ΔL	Change in length
A	Area
ν	Poisson's ratio
E	Young's modulus
G	Shear modulus
Q	First moment of area / statical moment of area
I	Area moment of inertia / second moment of area
J	Second polar moment of area
k_s	Linear spring constant
k_t	Torsional spring constant
U	Strain energy
W_E	External work
K_E	Kinetic energy
H_E	Heat absorbed
Π	Total potential energy
w	Shape function
κ	Curvature

1 Introduction

A very important aspect of aerospace structures is the way they fail. No one wants to lose a wing at 35,000ft. Understanding under what loads and in what manner the structures will fail can help us when designing fail-safe and damage-tolerant designs. We will broadly investigate two different failure modes, buckling and yielding.

2 Buckling - Compression Failure

Buckling is the abrupt change of shape of a structural element under some loading condition. In this course, we will investigate buckling failures due to compressive loads. An example of this can be seen when a soda can is crushed and changes shape (Fig. 1).

Figure 1: Soda Can Buckling Failure (Wright via American Physical Society)

When dealing in buckling failures, there are certain quantities we are typically interested in that I alluded to before. First, we care about the buckling load (P_{cr}). This is the critical compressive load under which the structure will first buckle. Second is the buckling mode (or mode shape). This describes how the structure will fail, aka the shape the structure will take post buckling.

To solve for these quantities, we will employ two methods: the Analytical Method and the Rayleigh-Ritz Method.

2.1 Analytical Method

The **Analytical Method** uses Euler-Bernoulli beam theory and stability theory to analyze the buckling behavior of beams.

From Euler-Bernoulli beam theory, we have a governing differential equation (Eq. 1) which describes a beam under compressive axial loading.

$$\frac{d^4w}{dx^4} + \lambda^2 \frac{d^2w}{dx^2} = g(x) \tag{1}$$

Note that we define $\lambda = \sqrt{\frac{P}{EI}}$, and $g(x)$ is a function which represents the contribution of external distributed loads on the structure. Most importantly, $w(x)$ is the shape function which is used to describe the displacement field of a structure. The shape function is a valuable tool because every displacement and load in a structure can be written as a function of it. See Tab. 1 for examples. Note that the function for the internal shear force is written here to include the contribution due to an external compressive load P.

Shape Function Representation	Physical Meaning
$w(x)$	Linear Displacement
$w'(x)$	Rotational Displacement
$M(x) = -EIw''(x)$	Bending Moment
$V(x) = -EIw'''(x) - Pw'(x)$	Shear Force

Table 1: Shape Function Meanings

Using the governing differential equation (Eq. 1), we can solve for the **General Shape Function** for a beam given by Eq. 2.

$$w(x) = A\sin(\lambda x) + B\cos(\lambda x) + Cx + D + f(x) \tag{2}$$

While we have this general shape function describing the bending of a beam, we still do not have all the information we need to solve the problem. We do not know values for the coefficients A, B, C, D, or $f(x)$. To solve, we will use the boundary conditions of the problem. The boundary conditions can be split into two categories: displacement boundary conditions and loading boundary conditions.

Displacement Boundary Conditions account for the information we know about the displacement field of the structure. They take advantage of the physical interpretation of the shape function as a linear and rotational displacement. At joint locations, we know the displacement field is fixed in some manner. Thus, we can use this information to help us solve for the shape function coefficients. In particular, we can use the equations given in Eq. 3. If the beam is fixed such that it cannot displace linearly at location x, we can say that $w(x) = 0$. If the beam is fixed such that it cannot displace rotationally at location x, we can say that $w'(x) = 0$.

$$\begin{aligned} w(x) &= 0 \\ w'(x) &= 0 \end{aligned} \tag{3}$$

Loading Boundary Conditions account for the information we know about the loading of a structure, particularly the bending moments and shear forces. In particular, we can use the equations given in Eq. 4.

$$\begin{aligned} M(x) &= -EIw''(x) \\ V(x) &= -EIw'''(x) - Pw'(x) \end{aligned} \tag{4}$$

Finally, we also have some idea of what $f(x)$ will be. $f(x)$ is a contribution to the shape function that results from a distributed load. If we are given a distributed load over our beam $q(x)$, we can rewrite $f(x)$ using Eq. 5. Otherwise, $f(x) = 0$.

$$f(x) = \frac{q(x)x^2}{2\lambda^2 EI} = \frac{q(x)x^2}{2P} \tag{5}$$

Not considering $f(x)$ since that is trivial to calculate, we know we will need 4 boundary conditions to solve for 4 unknown coefficients (A, B, C, and D). Once we've identified 4 boundary conditions, we can rewrite them in matrix form with Eq. 6.

$$[M] \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \vec{b} \tag{6}$$

Now there is also another unknown we are trying to solve for: the critical buckling load P_{cr}. We can solve for this by setting the characteristic equation of $[M]$ equal to zero as seen in Eq. 7. Also recall that $\lambda_{cr} = \sqrt{\frac{P_{cr}}{EI}}$

$$det([M(\lambda_{cr})]) = 0 \tag{7}$$

Finally, we can use P_{cr} to solve for the unknown coefficients (A, B, C, and D) by substituting the value of P_{cr} back into the system of equations. Unfortunately, we cannot explicitly solve for every unknown coefficient, and one will be left independent (or unknown). However, the other three can be written as a function of this one independent coefficient. With these coefficients identified, we can now write the buckling mode (or mode shape) as $w(x)$.

Changing the value of the independent coefficient will not change the "shape" of the mode shape. Instead, it will simply change the magnitude of mode shape, which we can adjust to fit real world displacements later on. An example of how the mode shape can vary with different values of the independent coefficient is depicted in Fig. 2. We can see that the independent coefficient acts as a scaling factor. While the general shape remains the same, the displacement at buckling can be adjusted with the independent coefficient.

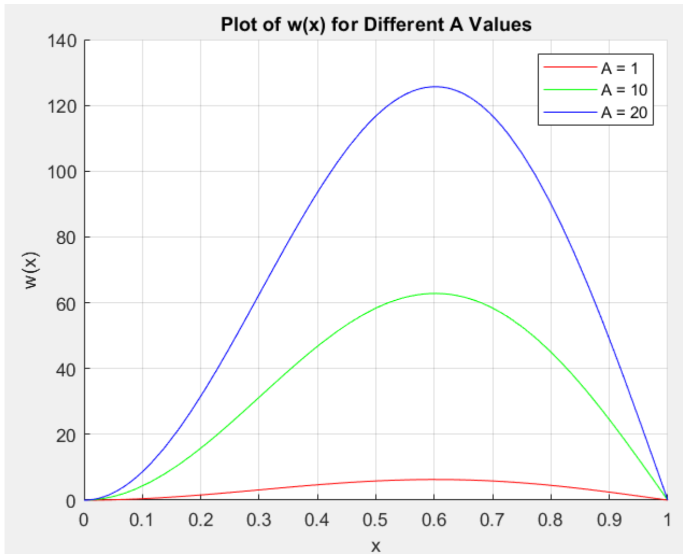

Figure 2: How Mode Shape can Vary with Independent Coefficient

Another important concept of note is the **Symmetry Condition**. The symmetry condition allows us to simplify a beam buckling problem by making the assumption that the displacement and loading in a beam is symmetric about some midpoint. Thus, we can add a specific boundary condition given by Eq. 8 (assuming the symmetry is about $x = L/2$).

$$w'(L/2) = 0 \tag{8}$$

Let's try and outline this whole process in a succinct manner.

Applying Analytical Buckling Method:

1. Write general shape function given by Eq. 9

$$w(x) = A\sin(\lambda x) + B\cos(\lambda x) + Cx + D + f(x) \tag{9}$$

2. Do we have a distributed load ($q(x)$)? If yes, rewrite $f(x)$ using Eq. 10. Otherwise, set $f(x) = 0$

$$f(x) = \frac{q(x)x^2}{2\lambda^2 EI} = \frac{q(x)x^2}{2P} \tag{10}$$

3. Identify four boundary conditions using both displacement and loading boundary conditions (Tab. 2)

Shape Function Representation	Physical Meaning
$w(x)$	Linear Displacement
$w'(x)$	Rotational Displacement
$M(x) = -EIw''(x)$	Bending Moment
$V(x) = -EIw'''(x) - Pw'(x)$	Shear Force

Table 2: Shape Function Meanings

4. Rewrite boundary conditions in matrix form using the relationship given in Eq. 11

$$[M]\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \vec{b} \tag{11}$$

5. Solve for the critical compressive buckling load (P_{cr}), using Eqs. 12 and 13

$$det([M(\lambda_{cr})]) = 0 \tag{12}$$

$$\lambda_{cr} = \sqrt{\frac{P_{cr}}{EI}} \tag{13}$$

6. Solve for A, B, C, and D by substituting the value for P_{cr} into the boundary condition equations. Note that one of these coefficients will be unknown (independent coefficient), and we can make solving easier by setting this unknown coefficient equal to 1

Let's run through a quick example to solidify the concept. Find the critical compressive buckling load and mode shape from Fig. 3.

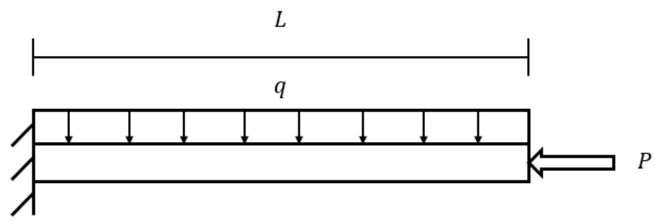

Figure 3: Analytical Buckling Example

First, we can write the general shape function given by Eq. 14.

$$w(x) = A\sin(\lambda x) + B\cos(\lambda x) + Cx + D + f(x) \tag{14}$$

Since we have a distributed load, we can represent $f(x)$ with Eq. 15.

$$f(x) = \frac{qx^2}{2P} \tag{15}$$

We need to identify 4 boundary conditions in order to solve for the 4 unknown coefficients A, B, C, and D. Let's start with displacement boundary conditions. Since one end of the beam is fixed, we know that the linear and rotational displacements at $x = 0$ will be 0 (Eq. 16).

$$\begin{aligned} w(0) &= 0 \\ w'(0) &= 0 \end{aligned} \tag{16}$$

Investigating the loading boundary displacements, we know that the moment and shear force at the free end will be 0. We can write this explicitly using Eq. 17.

$$\begin{aligned} M(L) &= EIw''(L) = 0 \\ V(L) &= EIw'''(L) + Pw'(L) = 0 \end{aligned} \tag{17}$$

We can express the derivatives using Eq. 18.

$$\begin{aligned} w'(x) &= A\lambda\cos(\lambda x) - B\lambda\sin(\lambda x) + C + \frac{qx}{P} \\ w''(x) &= -A\lambda^2\sin(\lambda x) - B\lambda^2\cos(\lambda x) + \frac{q}{P} \\ w'''(x) &= -A\lambda^3\cos(\lambda x) + B\lambda^3\sin(\lambda x) \end{aligned} \tag{18}$$

And the boundary conditions can be written explicitly using Eqs. 19 - 22.

$$B + D = 0 \tag{19}$$

$$A\lambda + C = 0 \tag{20}$$

$$A\sin(\lambda L) + B\cos(\lambda L) = \frac{q}{EI\lambda^4} \tag{21}$$

$$C = -\frac{Lq}{EI\lambda^2} \tag{22}$$

We can now rewrite the boundary conditions in matrix form as seen with Eqs. 23 and 24.

$$[M] = \begin{bmatrix} 0 & 1 & 0 & 1 \\ \lambda & 0 & 1 & 0 \\ -EI\lambda^2\sin(\lambda L) & -EI\lambda^2\cos(\lambda L) & 0 & 0 \\ 0 & 0 & EI\lambda^2 & 0 \end{bmatrix} \tag{23}$$

$$[M]\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{q}{\lambda^2} \\ -Lq \end{bmatrix} \tag{24}$$

We can solve for the buckling load using the determinant of the boundary condition matrix $[M]$ (Eq. 25). We find the buckling load as Eq. 27.

$$det([M]) = E^2I^2\lambda^5\cos(\lambda L) = 0 \tag{25}$$

$$\lambda_{cr} = \frac{\pi}{2L} \tag{26}$$

$$P_{cr} = \frac{EI\pi^2}{4L^2} \tag{27}$$

We can solve for the 4 coefficients by solving the boundary conditions and determining the values in Eq. 28. In this particular example, we get an explicit solution. This is often not the case, however.

$$\begin{aligned} A &= \frac{Lq}{EI\lambda^3} \\ B &= -\frac{q(L\lambda\sin(L\lambda) - 1)}{EI\lambda^4\cos(L\lambda)} \\ C &= -\frac{Lq}{EI\lambda^2} \\ D &= \frac{q(L\lambda\sin(L\lambda) - 1)}{EI\lambda^4\cos(L\lambda)} \end{aligned} \tag{28}$$

Substituting these values for the coefficients into the shape function, we get Eq. 29.

$$w(x) = \frac{Lq}{EI\lambda^3}\sin(\lambda x) - \frac{q(L\lambda\sin(L\lambda) - 1)}{EI\lambda^4\cos(L\lambda)}\cos(\lambda x) - \frac{Lq}{EI\lambda^2}x + \frac{q(L\lambda\sin(L\lambda) - 1)}{EI\lambda^4\cos(L\lambda)} + \frac{qx^2}{EI\lambda^2} \tag{29}$$

Importantly, we know that $\cos(L\lambda_{cr}) = 0$. Because the coefficients B and D include $(\cos(L\lambda_{cr}))^{-1}$, we know that they will tend towards infinity. Thus, they will be much larger than the other components of the shape function, so we can neglect these other components. We can rewrite the shape function with Eq. 30.

$$w(x) = \frac{q(L\lambda\sin(L\lambda) - 1)}{EI\lambda^4}(1 - \cos(\lambda x)) \tag{30}$$

Finally, substituting in the critical buckling load, we find that the mode shape can be described with Eq. 31.

$$w(x) = \frac{16L^4q(\frac{\pi}{2} - 1)}{EI\pi^4}(1 - \cos(\frac{\pi x}{2L})) \tag{31}$$

2.2 Rayleigh-Ritz Method

The **Rayleigh-Ritz Method** uses energy method approximation techniques (like those seen in the Energy Methods note) to solve for the desired beam buckling quantities. Instead of using boundary conditions like in the analytical method, we will instead use the principle of minimum total potential energy to our advantage.

We start with some approximate shape function ($w(x)$) which we will use unknown coefficients to describe the displacement field of the beam. Based on the particular problem, we can write the total potential energy (Π) as a combination of the strain energy (U) and external work applied (W_E) as seen in Eq. 32.

$$\Pi = U - W_E \tag{32}$$

Once we have written the total potential energy as a function of the shape function, we can use the principle of minimum total potential energy to write a system of equations with respect to the unknown shape function coefficients (Eq. 33).

$$\frac{\partial \Pi}{\partial c_1} = 0, \quad \frac{\partial \Pi}{\partial c_2} = 0, \quad ... \quad \frac{\partial \Pi}{\partial c_n} = 0 \tag{33}$$

We can write this equation in matrix form as seen in Eq. 34.

$$[M]\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{b} \tag{34}$$

From this matrix form, we can solve for the critical buckling load P_{cr} using Eq. 35.

$$det([M]) = 0 \tag{35}$$

Finally, we can find the mode shape by finding the unknown coefficients with the relationship in 34 where we've substituted P_{cr} for P. Unfortunately, we don't have enough information to explicitly solve for both coefficients. However, we should be able to write the mode shape in terms of a single independent coefficient (see previous section for more details).

Let's try and outline this whole process in a succinct manner.

Applying Rayleigh-Ritz Method:

1. Get the shape function $w(x)$ (typically given)

2. Find the total potential energy using Eq. 36. Note some common strain energy and applied external work formulations are provided here

$$\Pi = U - W_E \tag{36}$$

$$U_{moment} = \frac{1}{2}\int_0^L EI\left(w''(x)\right)^2 dx \tag{37}$$

$$W_{E_{axial}} = \frac{1}{2}\int_0^L P(x)\left(w'(x)\right)^2 dx \tag{38}$$

3. Set the derivative of the total potential energy with respect to each unknown coefficient to zero as shown in Eq. 39

$$\frac{\partial \Pi}{\partial c_1} = 0, \quad \frac{\partial \Pi}{\partial c_2} = 0, \quad ... \quad \frac{\partial \Pi}{\partial c_n} = 0 \tag{39}$$

4. Rewrite the derivative system of equations in matrix form as seen in Eq. 40

$$[M]\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{b} \tag{40}$$

5. Solve for the buckling load using Eq. 41

$$det([M]) = 0 \tag{41}$$

6. Solve for unknown coefficients (c_1, ... c_n) by substituting the value for P_{cr} into the boundary condition equations. Note that one of these coefficients will be unknown (independent coefficient), and we can make solving easier by setting this unknown coefficient equal to 1

Let's solidify this process with an example. In fact, let's do the same problem from the analytical section. Find the critical buckling load and the mode shape from Fig. 4. Use the shape function given in Eq. 42.

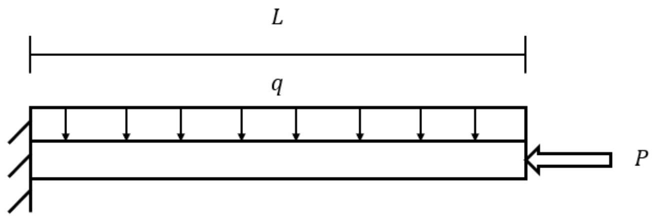

Figure 4: Rayleigh-Ritz Buckling Example

$$w(x) = c_1 x^3 + c_2 x^4 \tag{42}$$

First we are going to write the total potential energy of the system. We can write the strain energy in the system with Eq. 43, and the external work with the Eq. 44. The total potential energy can be expressed with Eq. 45

$$\begin{aligned} U &= \frac{1}{2}\int_0^L EI\left(w''(x)\right)^2 dx \\ &= \frac{6EIL^3(12L^2c_2^2 + 15Lc_1c_2 + 5c_1^2)}{5} \end{aligned} \tag{43}$$

$$\begin{aligned} W_E &= \frac{1}{2}\int_0^L P\left(w'(x)\right)^2 dx + \int_0^L qw(x)dx \\ &= \frac{L^5P(80L^2c_2^2 + 140Lc_1c_2 + 63c_1^2)}{70} + \frac{L^4q(5c_1 + 4Lc_2)}{20} \end{aligned} \tag{44}$$

$$\begin{aligned} \Pi &= U - W_E \\ &= \frac{6EIL^3(12L^2c_2^2 + 15Lc_1c_2 + 5c_1^2)}{5} \\ &\quad - \left(\frac{L^5P(80L^2c_2^2 + 140Lc_1c_2 + 63c_1^2)}{70} + \frac{L^4q(5c_1 + 4Lc_2)}{20}\right) \end{aligned} \tag{45}$$

We can then set the derivatives of the total potential energy with respect to the coefficients equal to zero as in Eq. 46.

$$\begin{aligned} \frac{\partial \Pi}{\partial c_1} &= 0 \\ \frac{\partial \Pi}{\partial c_2} &= 0 \end{aligned} \tag{46}$$

After this, we can represent the system of derivatives using a matrix relationship (Eq. 40). Matrix $[M]$ can be written as in Eq. 47.

$$[M] = \begin{bmatrix} -\frac{9PL^5}{5} + 12EIL^3 & -2PL^6 + 18EIL^4 \\ -2PL^6 + 18EIL^4 & -\frac{16PL^7}{7} + \frac{144EIL^5}{5} \end{bmatrix} \tag{47}$$

Similarly, $\vec{b}$ can be written with Eq. 48.

$$\vec{b} = \begin{bmatrix} \frac{L^4 q}{4} \\ \frac{L^5 q}{5} \end{bmatrix} \tag{48}$$

We can solve for the critical buckling load using Eq. 41. When solving this, we find two potential buckling loads given by Eq. 49.

$$\begin{aligned} P_{cr1} &= -\frac{3EI(2\sqrt{571} - 53)}{5L^2} \\ P_{cr2} &= \frac{3EI(2\sqrt{571} + 53)}{5L^2} \end{aligned} \tag{49}$$

To find the mode shape, we can substitute the buckling load back into $[M]$ and $\vec{b}$, and solve for the relationship between the coefficients (since we don't have enough information to solve for both c_1 and c_2) as in Eq. 50. We can do this for buckling load 1 and load 2 (Eq. 51).

$$[M(P_{cr})] \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \vec{b}(P_{cr}) \tag{50}$$

$$\begin{aligned} c_1(P_{cr1}) &= \frac{0.0016(25Lq - 1174.946EILc_2)}{EI} \\ c_1(P_{cr2}) &= \frac{0.0001(25Lq - 10294.946EILc_2)}{EI} \end{aligned} \tag{51}$$

With this ratio between coefficients, we can write the mode shape for buckling load 1 with Eq. 52, and the mode shape for buckling load 2 with Eq. 53.

$$w(x) = c_2 \left(\frac{0.0016(25Lq - 1174.946EIL)}{EI} x^3 + x^4 \right) \tag{52}$$

$$w(x) = c_2 \left(\frac{0.0001(25Lq - 10294.946EIL)}{EI} x^3 + x^4 \right) \tag{53}$$

3 Yielding - Tension Failure

Material Failure is defined as the loss of the load carrying capacity of a material. In other words when a material can no longer bear loads, it has failed. Defining when failure may occur often relies on a multitude of different theory approaches which perform more or less accurately given the context of the problem. There are also different classifications of failure for materials that are ductile or brittle.

1. **Ductile Materials** will deform and stretch plastically before fracture occurs. For these materials, the yield stress typically indicates the transition from elastic deformation to plastic deformation.
2. **Brittle Materials** will not deform plastically and instead fracture. For these materials, the yield stress typically indicates the failure of the material.

For isotropic materials, the most popular failure conditions are stress dependent. For this reason, we will cover the failure criteria of Tresca and von Mises. We will also determine the residual stresses left from plastically deforming a structure.

3.1 Tresca and von Mises

Tresca and von Mises criterias allow us to relate the principle stresses to the yield stress. Solving for the yielding stress based on either criteria will give the 1D stress state that will induce yielding in the material. Consequentially, the reverse is also true.

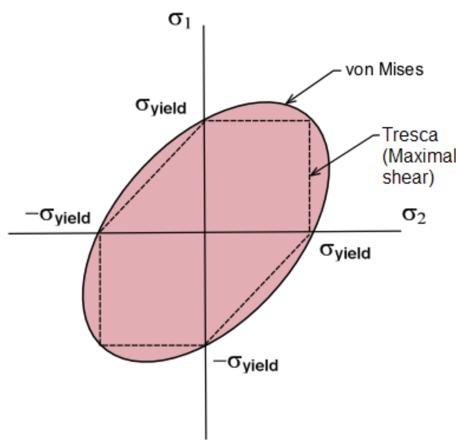

Figure 5: Relationship Between Tresca and von Mises Failure Criteras

The **Tresca Criteria** is typically more conservative than the von Mises criteria (relationship seen in Fig. 5). It argues that the yielding stress can be found when the maximum shear stress reaches a critical value dependent on the principle stresses. Using Tresca, the yield stress can be expressed using Eq. 54.

$$\sigma_T = max\{|\sigma_1 - \sigma_2|, |\sigma_1 - \sigma_3|, |\sigma_2 - \sigma_3|\} \tag{54}$$

The **von Mises Criteria** is typically more accurate than Tresca criteria. It argues that the yielding stress can be found based on the maximum distortion energy. Similarly to Tresca, this yield stress can be expressed based on the principle stresses (Eq. 55).

$$\sigma_{vM} = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2}{2}} \tag{55}$$

3.2 Residual Stress

Residual Stresses are the stresses left in a material after inelastic deformations occur. When plastic deformation occurs, the material will not elastically return to its original shape. Instead, the material has been permanently altered. This permanent alteration can lead to residual stresses.

We can calculate the residual stress in a few different ways, but it all boils down to one formula (Eq. 56). This formula states that the residual stresses can be calculated by taking the difference between the actual stresses in a material and subtracting the stress if the material behaved purely elastically.

$$\sigma_R = \sigma_{actual} - \sigma_{elastic} \tag{56}$$

Let's walk through an example to concretely illustrate this concept. In Fig. 7, we are given a truss structure composed of three beams: **AD**, **BD**, and **CD**. We assume they all have the same properties, and that P is applied to the point that all bars reach the yield stress, σ_y. We want to determine the residual stress left in the structure if the load P is removed.

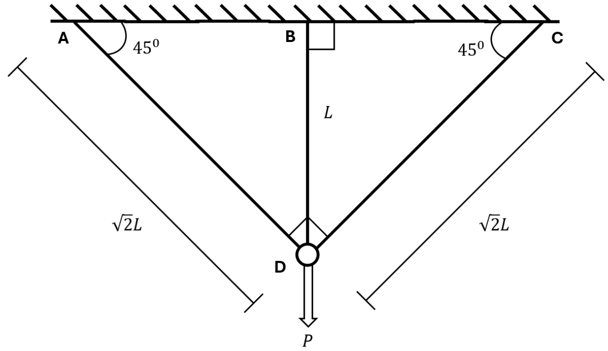

Figure 6: Yielding Example

To solve this problem, let us return to Eq. 56. To solve for the residual stress, we need to know the actual stress and the pure elastic stress. Based on the problem statement, we already know the actual stress of all the beams are σ_y (Eq. 57). Great, we're already half way there! Now, we need to use elastic stress-strain relationships to solve for the pure elastic stress.

$$\sigma_{AD}^{actual} = \sigma_{BD}^{actual} = \sigma_{CD}^{actual} = \sigma_y \tag{57}$$

Firstly, we need to determine what the load P actually is. We know that the actual stress in each bar is σ_y. Thus, we also know that the axial load each bar bears can be given Eq. 58.

$$F_{AD} = F_{BD} = F_{CD} = \sigma_y A \tag{58}$$

Sketching the FBD of the system (Fig. 7), we can use equilibrium equations to solve for P in Eq. 59.

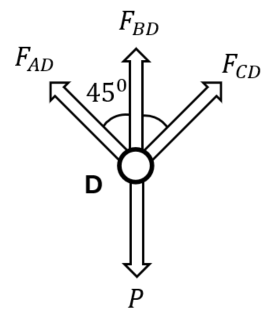

Figure 7: FBD of Yielding Structure

$$\begin{aligned} \sum F_y = 0 &= (F_{AD} + F_{CD})\cos(45^\circ) + F_{BD} - P \\ P &= (1 + \sqrt{2})\sigma_y A \end{aligned} \tag{59}$$

Now that we know what P is, we can find a relationship between all axial loads in the structure assuming purely elastic behavior. We know that the relationship between the elongation of the bars can be given by Eq. 60 using geometry.

$$\Delta L_{AD} = \Delta L_{CD} = \frac{\sqrt{2}}{2}\Delta L_{BD} \tag{60}$$

With this information, we can relate F_{AD}, F_{BD}, and F_{CD} using stress strain relationships as seen in Eqs. 61 and 62.

$$\begin{aligned} \sigma_{AD} &= E\varepsilon_{AD} = E\frac{\Delta L_{AD}}{L_{AD}} \\ F_{AD} &= EA\frac{\Delta L_{AD}}{L_{AD}} \\ F_{AD} &= F_{CD} = \frac{1}{2L}EA\Delta L_{BD} \end{aligned} \tag{61}$$

$$
\begin{aligned}
\sigma_{BD} &= E\varepsilon_{BD} = E\frac{\Delta L_{BD}}{L_{BD}} \\
F_{BD} &= EA\frac{\Delta L_{BD}}{L_{BD}} \\
F_{AD} &= F_{CD} = \frac{1}{L}EA\Delta L_{BD}
\end{aligned} \tag{62}
$$

And we can relate the axial loads to one another with Eq. 63.

$$F_{AD} = F_{CD} = \frac{1}{2}F_{BD} \tag{63}$$

Returning to the equilibrium equations, we can now solve for the axial loads for purely elastic deformations in terms of know quantities (Eq. 64) using the value for P we determined previously.

$$
\begin{aligned}
\sum F_y = 0 &= (F_{AD} + F_{CD})\cos(45^\circ) + F_{BD} - P \\
P &= (1 + \frac{\sqrt{2}}{2})F_{BD} \\
F_{BD} &= \frac{P}{(1 + \frac{\sqrt{2}}{2})} \\
F_{BD} &= \frac{(1 + \sqrt{2})\sigma_y A}{(1 + \frac{\sqrt{2}}{2})} \\
F_{BD} &= \sqrt{2}\sigma_y A \\
F_{AD} &= F_{CD} = \frac{\sqrt{2}}{2}\sigma_y A
\end{aligned} \tag{64}
$$

Finally, we can solve for the purely elastic stress in each bar with Eq. 65.

$$
\begin{aligned}
\sigma_{AD}^{elastic} &= \sigma_{CD}^{elastic} = \frac{\sqrt{2}}{2}\sigma_y \\
\sigma_{BD}^{elastic} &= \sqrt{2}\sigma_y
\end{aligned} \tag{65}
$$

Using Eq. 56, we now find the residual stresses in each beam as Eq. 66.

$$
\begin{aligned}
\sigma_{AD}^{residual} &= \sigma_{CD}^{residual} = (1 - \frac{\sqrt{2}}{2})\sigma_y \\
\sigma_{BD}^{residual} &= (1 - \sqrt{2})\sigma_y
\end{aligned} \tag{66}
$$

And with that, we've finished the problem!

A MATLAB Codes

```
% AA240
% Structural Failure: Analytical Method Example
% Author(s): Mark Paral

% Clear workspace
clc
clear all
close all

% Symbols
syms P lambda x A B C D E I q L

% Shape function
f_x = q*x^2 / (2*lambda^2*E*I);
w = A*sin(lambda*x) + B*cos(lambda*x) + C*x + D + f_x;

% Shape derivatives
wp = diff(w,x);
wpp = diff(wp,x);
wppp = diff(wpp,x);

% Boundary Conditions
eq1 = w == 0;
eq1 = subs(eq1,x,0);
eq2 = wp == 0;
eq2 = subs(eq2,x,0);
eq3 = E*I*wpp == 0;
eq3 = subs(eq3,x,L);
eq4 = E*I*wppp + P*wp == 0;
eq4 = subs(eq4,x,L);

% Show boundary conditions
disp("eq1")
simplify(collect(eq1, [A, B, C, D]))
disp("eq2")
simplify(collect(eq2, [A, B, C, D]))
disp("eq3")
simplify(collect(eq3, [A, B, C, D]))
disp("eq4")
simplify(collect(eq4, [A, B, C, D]))

% Solve for buckling load
```

```
[M,b] = equationsToMatrix([eq1,eq2,eq3,eq4],[A,B,C,D])
    ;
det(M)
eq5 = subs(det(M),P,lambda^2*E*I) == 0;
lambda_cr = solve(eq5,lambda);
P_cr = lambda_cr(6)^2*E*I;

% Solve for coefficients
b = subs(b,P,lambda^2*E*I);
M = subs(M,P,lambda^2*E*I);
eq6 = M * [A;B;C;D] == b;
[A_val,B_val,C_val,D_val] = solve(eq6,[A,B,C,D]);
disp("A:")
pretty(simplify(A_val))
disp("B:")
pretty(simplify(B_val))
disp("C:")
pretty(simplify(C_val))
disp("D:")
pretty(simplify(D_val))

% Final mode shape
pretty(simplify(subs(simplify((B_val*cos(lambda*x) +
    D_val) ...
     * cos(L*lambda)),lambda,lambda_cr(6))))
```

```
% AA240
% Structural Failure: Rayleigh-Ritz Method Example
% Author(s): Mark Paral

% Clear workspace
clc
clear all
close all

% Symbols
syms P E I L x q c1 c2

% Shape function
w = c1*x^3 + c2*x^4;

% Shape derivatives
wp = diff(w,x);
wpp = diff(wp,x);
wppp = diff(wpp,x);

% Total potential energy
U = 0.5*int(E*I*wpp^2,x,[0,L]);
WE = 0.5*int(P*wp^2,x,[0,L]) + int(q*w,x,[0,L]);
Pi = U - WE;
disp("Pi:")
pretty(simplify(Pi))

% Partial derivatives
eq1 = diff(Pi,c1) == 0;
eq2 = diff(Pi,c2) == 0;

% Get matrix representation
[M,b] = equationsToMatrix([eq1,eq2],[c1,c2]);
disp("M:")
pretty(simplify(M))
disp("b:")
pretty(simplify(b))

% Find the critical buckling load
eq3 = det(M) == 0;
P_cr = solve(eq3,P);
disp("P_cr:")
pretty(simplify(P_cr))
P_cr1 = P_cr(1);
P_cr2 = P_cr(2);
```

```
% Find the mode shape
M1 = subs(M,P,P_cr1);
b1 = subs(b,P,P_cr1);
M2 = subs(M,P,P_cr2);
b2 = subs(b,P,P_cr2);

eq3 = M1*[c1;c2]==b1;
c1_val_1 = solve(eq3(1),c1);
eq4 = M2*[c1;c2]==b2;
c1_val_2 = solve(eq4(1),c1);

disp("Coefficients:")
disp("- Case 1 c1:")
pretty(vpa(c1_val_1))
disp("- Case 2 c1:")
pretty(vpa(c1_val_2))
```